THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2060B Mathematical Analysis II (Spring 2017)
HW4 Solution

Yan Lung Li

1. (P.215 Q2)

h is clearly bounded on $[0,1]$. Applying Theorem 1.8 of the lecture note 1 P.3, it suffices to show that there exists $\epsilon_0 > 0$ such that for all partition $P := a = x_0 < x_1 < ... < x_n = b$ on $[0,1]$, we have

$$U(h,P) - L(h,P) \geq \epsilon_0$$

Let $\epsilon_0 = 1$, then for all partition $P := a = x_0 < x_1 < ... < x_n = b$ on $[a,b]$. For each $1 \leq i \leq n$, since $\mathbb{Q} \cap [0,1]$ is dense in $[0,1]$, there exists $(y_m^{(i)})_{m=1}^{\infty} \subseteq \mathbb{Q} \cap [x_{i-1}, x_i]$ such that $y_m^{(i)} \to x_i$ as $m \to \infty$. Since $h(x) \leq x_i + 1$ on $[x_{i-1}, x_i]$ by definition, we have

$$M_i(h,P) = x_i + 1$$

On the other hand, since $(\mathbb{R}\backslash\mathbb{Q}) \cap [0,1]$ is dense in $[0,1]$, $(\mathbb{R}\backslash\mathbb{Q}) \cap [x_{i-1}, x_i] \neq \phi$, and hence $h(z_i) = 0$ for some $z_i \in (\mathbb{R}\backslash\mathbb{Q}) \cap [x_{i-1}, x_i]$. Since $h(x) \geq 0$ on $[x_{i-1}, x_i]$ by definition, we have

$$m_i(h,P) = 0$$

Therefore,

$$\begin{aligned} U(h,P) - L(h,P) &= \sum_{i=1}^{n} \omega_i(h,P)\Delta x_i \\ &= \sum_{i=1}^{n} (x_i + 1)(x_i - x_{i-1}) \\ &\geq \sum_{i=1}^{n} (x_i - x_{i-1}) \\ &= x_n - x_0 = 1 = \epsilon_0 \end{aligned}$$

Therefore, h is not integrable on $[0,1]$.

2. (P.215 Q8)

We prove by contradiction: suppose there exists $x_0 \in [a,b]$ such that $f(x_0) \neq 0$, i.e. $f(x_0) > 0$. Since f is continuous on $[a,b]$, using its continuity at x_0 with $\epsilon = \dfrac{f(x_0)}{2} > 0$, there exists $\delta > 0$ such that $\forall x \in V_\delta(x_0) \cap [a,b]$, $|f(x) - f(x_0)| < \epsilon$, and hence

$$\begin{aligned} f(x) &= f(x_0) - (f(x_0) - f(x)) \\ &> f(x_0) - \epsilon \\ &= \frac{f(x_0)}{2} > 0 \end{aligned}$$

Now, since f is non-negative,

$$\int_a^b f \geq \int_{[a,b]\cap V_\delta(x_0)} f$$

and by construction, we have

$$\begin{aligned}\int_{[a,b]\cap V_\delta(x_0)} f &\geq \frac{f(x_0)}{2}\cdot(\text{length of }([a,b]\cap V_\delta(x_0)) \\ &\geq \frac{f(x_0)}{2}\cdot\delta > 0\end{aligned}$$

These imply

$$\int_a^b f > 0$$

which contradicts to the assumption that $\int_a^b f = 0$.

Therefore, $f(x) = 0$ for all $x \in [a,b]$.

3. (P.215 Q8)

Define $h(x) = f(x) - g(x)$ on $[a,b]$, then h is continuous on $[a,b]$ (and hence Riemann integrable by Prop. 1.11 in Lecture note 1 P.5) and $\int_a^b h = \int_a^b f - \int_a^b g$ (by Prop. 1.7 of lecture note 1 P.3)$= 0$.

Now we prove by contradiction: suppose on the contrary for all $c \in [a,b]$, $f(c) \neq g(c)$, i.e. $h(c) \neq 0$. Since h is continuous on $[a,b]$, by Intermediate Value Theorem, either (i) $h(x) > 0$ for all $x \in [a,b]$ or (ii) $h(x) < 0$ all $x \in [a,b]$.

Case (i): applying the result of Q8 (since h is non-negative on $[a,b]$ and $\int_a^b h = 0$), we must have $h(x) = 0$ for all $x \in [a,b]$, which is a contradiction.

Case (ii) Let $k(x) = -h(x)$ on $[a,b]$. Then apply case (i) to $k(x)$ to derive a contradiction.

Therefore, both leads to contradiction. Hence there exists $c \in [a,b]$ such that $f(c) = g(c)$.